

Write all responses on separate paper. Show your work in detail for credit. No calculators.

1. (18 points) Calculate $\frac{dy}{dx}$

(a) $y = \frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[4]{x^3}}$

SOLN: $-\frac{1}{3}x^{-4/3} + \frac{3}{4}x^{-7/4} = \frac{1}{12}x^{-7/4}(4x^{5/12} - 9)$

(b) $y = \frac{\sin(2x)}{1 - \tan(2x)}$

SOLN: $\frac{2 \cos(2x)(1 - \tan(2x)) + 2 \sin(2x) \sec^2(2x)}{(1 - \tan(2x))^2} = \frac{2 \cos(2x) - 2 \sin(2x) + 2 \sin(2x)(\tan^2(2x) + 1)}{(1 - \tan(2x))^2}$
 $= \frac{2 \cos(2x) + 2 \sin(2x) \tan^2(2x)}{(1 - \tan(2x))^2}$

(c) $y = e^{-x^2} \sin(3x)$

SOLN: $u \frac{dv}{dx} + v \frac{du}{dx} = 3e^{-x^2} \cos(3x) - 2xe^{-x^2} \sin(3x) = e^{-x^2}(3 \cos(3x) - 2x \sin(3x))$

(d) $\arcsin(\sqrt{x})$

SOLN: $\frac{du}{dx} \frac{d}{du} \arcsin(u) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1-u^2}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x(1-x)}}$

(e) $y = \cosh(x^2)10^{-x^2} \Rightarrow y' = 2x(\sinh(x^2)10^{-x^2} - \ln 10 \cdot 10^{-x^2} \cosh(x^2))$

(f) $\ln \left| \frac{1+x^2}{1-x^2} \right|$ SOLN: Long division gives $\frac{1+x^2}{1-x^2} = -1 + \frac{2}{1-x^2}$

$\frac{du}{dx} \frac{d}{du} \ln(u) = \frac{d}{dx} \left(-1 + \frac{2}{1-x^2} \right) \frac{1}{u} = \frac{4x}{(1-x^2)^2} \frac{1-x^2}{1+x^2} = \frac{4x}{1-x^4}$

2. (10 points) Use the **definition** of the derivative (that is, $f'(x) \equiv \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$) to compute $\frac{d}{dx} \sec(x)$

Hint: $\lim_{h \rightarrow 0} \frac{\sin h}{h}$ and $\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h}$

SOLN: $\lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x) - \cos(x+h)}{h \cos(x) \cos(x+h)} = \lim_{h \rightarrow 0} \frac{\cos(x) - \cos(x) \cos(h) + \sin(h) \sin(x)}{h \cos(x) \cos(x+h)}$
 $= \lim_{h \rightarrow 0} \frac{1}{\cos(x+h)} \frac{1 - \cos(h)}{h} + \frac{\sin(x)}{\cos(x) \cos(x+h)} \frac{\sin(h)}{h} = \frac{\sin(x)}{\cos^2(x)} = \sec(x) \tan(x)$

3. (10 points) Find an equation for the line tangent to the curve given by $xy^2 + yx^2 = 6$ at the point $(1, -3)$

SOLN: Assume y is a function of x and differentiate wrt x :

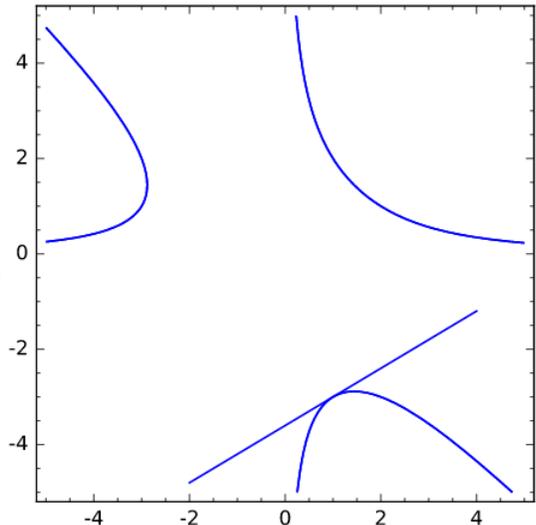
$y^2 + 2xy \frac{dy}{dx} + 2xy + x^2 \frac{dy}{dx} = 0 \Leftrightarrow$

$(2xy + x^2) \frac{dy}{dx} = -y^2 - 2xy$, but we don't have to solve for $\frac{dy}{dx}$ before

plugging in $(1, -3)$, so $\Rightarrow -5 \frac{dy}{dx} = -9 + 6 = -15 \Leftrightarrow \frac{dy}{dx} = \frac{3}{5}$, so an

equation of the tangent line is $y + 3 = \frac{3}{5}(x - 1)$

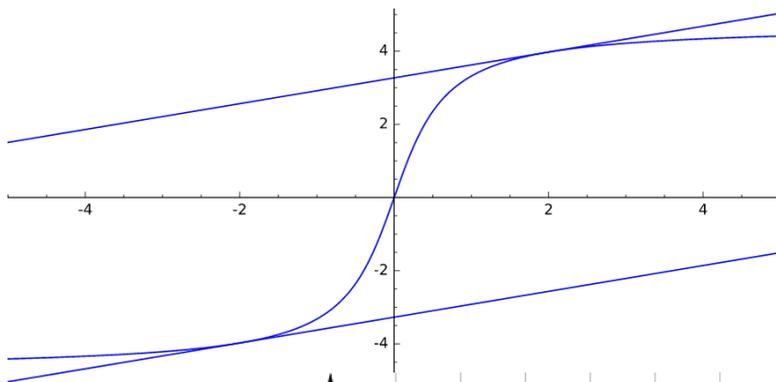
Let's illustrate:



4. (10 points) Find two points on the curve $y = 3 \arctan(2x)$ where the line tangent to the curve is parallel to $17y - 6x = 0$.

$$\text{SOLN: } y' = \frac{6}{1+4x^2} = \frac{6}{17} \Leftrightarrow x = \pm 4$$

So the points are $(-2, -3 \arctan(4))$ and $(2, 3 \arctan(4))$. Here's an illustration of this:



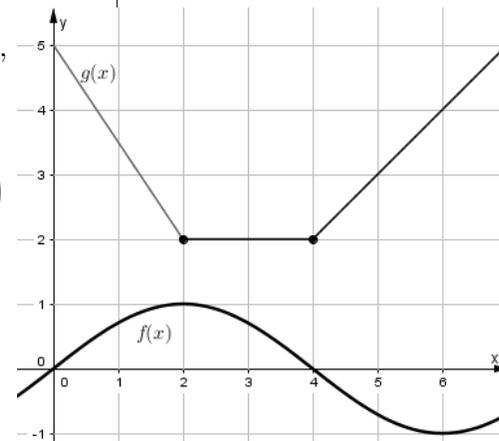
5. (12 points) If f and g are the functions whose graphs are shown, where $f(x)$ is sinusoidal function, $f(x) = \sin\left(\frac{\pi x}{4}\right)$.

Let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$ and $C(x) = f(g(x))$. Find

$$(a) P'(1) = f'(1)g(1) + f(1)g'(1) = \frac{7\sqrt{2}\pi}{16} - \frac{3\sqrt{2}}{4} = \frac{\sqrt{2}}{4} \left(\frac{7\pi}{4} - 3 \right)$$

$$(b) Q'(3) = \frac{g(3)f'(3) - g'(3)f(3)}{(g(3))^2} = \left(-\frac{\sqrt{2}\pi}{4} - 0 \right) \frac{1}{4} = -\frac{\sqrt{2}\pi}{16}$$

$$(c) C'(6) = f'(g(6)) \cdot g'(6) = f'(4) \cdot 1 = -\frac{\pi}{4}$$



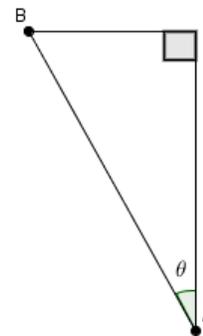
6. (8 points) The volume of a sphere is increasing at a rate of 4 cubic meters per hour. How fast is the surface area increasing when the radius 50 cm?

SOLN: The sphere's volume is $V = \frac{4}{3}\pi r^3$ so $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4 \frac{m^3}{hr} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi r^2} \frac{m}{hr}$. The surface area is $A = 4\pi r^2$

so $\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = \frac{8}{r}$. So when $r = 50\text{cm} = \frac{1}{2}m$, $\frac{dA}{dt} = 4 \frac{m^2}{hr}$

7. (10 points) Two students A and B are walking on straight roads that meet at right angles. Student A moves away from the intersection at 1 m/sec and student B approaches that intersection at 2 m/sec as shown in the figure. At what rate is the angle θ changing when A is 20 m from the intersection and B is 10 m from the intersection? Express your answer in radians per second.

Hint: To work this problem, it helps to introduce variables to represent the lengths of the sides and then express $\tan(\theta)$ in terms of those variables.



SOLN: Let x = the top edge and y = the right edge of the triangle. Then $\tan \theta = \frac{x}{y}$ and $\frac{d}{dt} \tan \theta = \sec^2 \theta \frac{d\theta}{dt} =$

$$\frac{y^2}{x^2 + y^2} = \frac{400 \frac{d\theta}{dt}}{500 \frac{d\theta}{dt}} = \frac{y \cdot dx/dt - x \cdot dy/dt}{y^2} = \frac{20(-2) - 10(1)}{400} \rightarrow \frac{d\theta}{dt} = \frac{5}{4} \cdot \frac{-5}{40} = \frac{-5 \text{ rad}}{32 \text{ sec}}$$

8. (10 points) A population of plankton experiences natural growth. On day zero the population has a mass of 4 grams. On day 10 the population has a mass of 100 grams. Find an expression for the population mass on day 15.

SOLN: $P(t) = 4e^{rt}$ gives the population model with the right starting population. Then $P(10) = 4e^{10r} = 100$ so $r = \frac{\ln 25}{10}$ so $P(15) = 4e^{1.5 \ln 25} = 4(25)^{1.5}$ grams after 15 days.

9. (12 points) Find the linearization of the function $f(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$

SOLN: $f'(x) = \frac{1}{3}(1+x)^{-2/3}$ so $f'(0) = \frac{1}{3}$ and the linearization is $y = f(0) + \frac{1}{3}(x-0) = 1 + \frac{1}{3}dx$. Thus

$$\sqrt[3]{0.95} \approx 1 - 0.05/3 = \frac{59}{60} \text{ and } \sqrt[3]{1.1} \approx 1 + 0.1/3 = \frac{31}{30}$$