

1. (12 points) The table below shows daylight hours in Palm Desert for the year 2015
 –table deleted–

(a) Use the table to find the average rate of change in daylight hours per day between May 1 and May 31 of this year. Be sure to specify the units of measure for this rate of change.

SOLN: The change in daylight hours is 42 minutes or $\frac{42}{60} = 0.7$ hours. This is over a period of 30 days so the average rate of change in the merry month of May is $\frac{0.7\text{hours}}{30 \text{ days}} \approx 0.023$ hours per day, equivalent to 1.4 minutes/day.

(b) What is your best approximation, based on this table, for rate of change on May 31?

SOLN: The average rate of change from May 30 to May 31 is 1 minute per day and that's the same as the average rate of change between May 31 and June 1 (at least to the level of accuracy available to us by this table.) So the best approximation to the average rate of change on May 31 is 1 minute of daylight per day, or $\frac{1}{60} = 0.01\bar{6}$ hours of daylight per day. We could also call this $0.00069\bar{4}$ days/day where the days cancel and it's just a pure number.

(c) Is it possible to find an instantaneous rate of change for May 31? Explain.

The daylight hours in Palm Desert do not change on a continuum, rather they come in discrete chunks per day. Therefore shrinking the denominator to zero (as is required for an instantaneous rate of change) is not possible. However, if you think of the length of a day as being distributed over the latitude of Palm Desert (about 34 degrees North) then we could talk about a per meter*day rate of change? Hmmm.

2. (12 points) Find the limit. Give baby steps and justify each step with a property of limits.

(a) $\lim_{x \rightarrow 1} \sqrt[3]{x - x^3}$

SOLN: Because the cube root and cubic polynomial functions are both continuous throughout their domains, their composition is also continuous, and since 1 is in the domain of $f(x) = x - x^3$ and $f(1) = 0$ is in the domain of $g(x) = \sqrt[3]{x}$ so we can compute $\lim_{x \rightarrow 1} \sqrt[3]{x - x^3} = \sqrt[3]{\lim_{x \rightarrow 1} x - x^3} = \sqrt[3]{0} = 0$.

(b) $\lim_{x \rightarrow 0^+} \frac{\lfloor x \rfloor}{x}$ where $\lfloor x \rfloor \equiv$ the greatest integer less than x .

SOLN: Since $\lfloor x \rfloor = 0$ for $x \in [0, 1)$, $\lim_{x \rightarrow 0^+} \frac{\lfloor x \rfloor}{x} = \lim_{x \rightarrow 0^+} \frac{0}{x} = 0$

(c) $\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 8x - 1} \right)$

SOLN: $\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 8x - 1} \right) = \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 8x - 1} \right) \frac{x + \sqrt{x^2 + 8x - 1}}{x + \sqrt{x^2 + 8x - 1}}$
 $= \lim_{x \rightarrow \infty} = \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 8x - 1)}{x + \sqrt{x^2 + 8x - 1}} = \lim_{x \rightarrow \infty} \frac{-8x + 1}{x + \sqrt{x^2 + 8x - 1}} = \lim_{x \rightarrow \infty} \frac{-8 + \frac{1}{x}}{1 + \sqrt{1 + 8/x - 1/x^2}} = \frac{-8}{1 + \sqrt{1}} = -4$

Also $\lim_{x \rightarrow \infty} \left(x - \sqrt{(x+4)^2 - 17} \right) =$
 $\lim_{x \rightarrow \infty} \left(x - (x+4)\sqrt{1 - \frac{17}{(x+4)^2}} \right)$

Note that graph in sagemath.org produced by
`f=x-sqrt(x^2+8*x-1)`
`plot(f, (x, 1, 100))`
 confirms this.

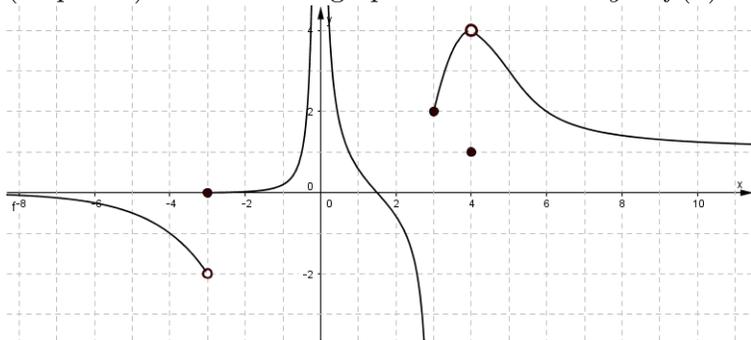
3. (12 points) Give a formal epsilon-delta definition proof that $\lim_{x \rightarrow 3} \frac{x}{3} + 1 = 2$

SOLN: Let $\delta = 3\epsilon$. Then $|x - 3| < \delta \Leftrightarrow |x - 3| < 3\epsilon \Leftrightarrow \left| \frac{x}{3} - 1 \right| < \epsilon \Leftrightarrow \left| \left(\frac{x}{3} + 1 \right) - 2 \right| < \epsilon \Leftrightarrow |f(x) - f(3)| < \epsilon$

4. (12 points) Find the smallest value of N so that if $x > N$ then $\ln(x) > 100$

SOLN: We want $|\ln(x)| > 100 \Leftrightarrow x > \ln(100)$ so, since $\ln(x)$ is an increasing function of x , $N = \ln(100)$ is the smallest value.

5. (16 points) Consider the graph of the function $y = f(x)$ shown below:



- (a) Find each limit, or explain why it does not exist. SOLNS:

i. $\lim_{x \rightarrow -3^-} f(x) = -2$ ii. $\lim_{x \rightarrow -3^+} f(x) = 0$ iii. $\lim_{x \rightarrow 3^+} f(x) = 2$ iv. $\lim_{x \rightarrow 3^-} f(x) = -\infty$

- (b) State the equations of the horizontal asymptotes suggested by the graph.

SOLN: It appears as though $y = 0$ (as $x \rightarrow -\infty$) and $y = 1$ (as $x \rightarrow \infty$) are horizontal asymptotes.

- (c) State the equations of the vertical asymptotes suggested by the graph.

SOLN: It appears as though $x = 0$ (as $|x| \rightarrow 0$) and $x = 3$ (as $x \rightarrow 3^-$) are vertical asymptotes.

- (d) At what numbers is f discontinuous? Explain.

SOLN: f has a jump discontinuity at $x = -3$ a vertical asymptote at $x = 3$ and a removable discontinuity at $x = 4$. It also appears to have a vertical asymptote at $x = 0$ in which case $x = 0$ is not in the domain.

6. (10 points) Let

$$f(x) = \begin{cases} x - 1 & : x \leq 2 \\ ax^2 + 3 & : 2 < x \leq 3 \\ x^3 + b & : 3 < x \end{cases}$$

Find values of a and b so that f is a continuous function.

SOLN: First observe that $\lim_{x \rightarrow 2^-} f(x) = 1$ and so continuity requires $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} ax^2 + 3 = 4a + 3 = 1$, so

$$a = -\frac{1}{2}. \text{ Now } \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} -\frac{x^2}{2} + 3 = -\frac{3}{2} \text{ so we require } \lim_{x \rightarrow 3^+} f(x) = 27 + b = -\frac{3}{2} \Leftrightarrow b = -\frac{57}{2} = -28.5$$

7. (12 points) Consider $f(x) = \frac{1}{x+1}$

- (a) Use the definition of the derivative to find the derivative function $f'(x)$

$$\begin{aligned} \text{SOLN: } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{x+1 - (x+h+1)}{(x+h+1)(x+1)} = \lim_{h \rightarrow 0} \frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+1)^2} \end{aligned}$$

- (b) Find an equation for the line tangent to the function at $x = 2$

$$\text{SOLN: The slope is } \frac{-1}{(2+1)^2} = -\frac{1}{9} \text{ and the point slope formula for a line yields } y - \frac{1}{3} = -\frac{1}{9}(x - 2)$$

8. (14 points) Use the intermediate value theorem to show that the equation $3^x = x^3$ has a solution for $0 < x < 2.5$. First state the Intermediate Value Theorem, then show precisely how the premise is satisfied and what conclusion follows. *Hint:* $\sqrt{3} \approx 1.732$

SOLN: The Intermediate Value Theorem states that if $f(x)$ is continuous on $[a, b]$ and n is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = n$.

In this case, let $f(x) = x^3 - 3^x$. As a difference of everywhere continuous functions, this function is continuous everywhere. In particular, it's continuous on $[0, 2.5]$ Now $f(0) = -1$ and $f(2.5) = \left(\frac{5}{2}\right)^3 - 3^{2.5} = \frac{125}{8} - 9\sqrt{3} \approx 15.625 - 9(1.73) = 15.625 - 15.588 > 0$ So f , a continuous function, goes from negative to positive on $[0, 2.5]$ and by the IVT there exists $c \in [0, 2.5]$ such that $f(c) = 0$ and c is then a solution to the equation.