

## Math 1A - Take Home Project for Newton's Method – Fall '10

Write your answers to following questions using appropriate exposition so that your answer can be read as an essay. Show all your work and explain what you're doing as if you're trying to teach a classmate what's going on.

1. Suppose  $f$  is a quadratic polynomial with distinct real zeros. Denote these zeros by  $r_1$  and  $r_2$  with  $r_1 < r_2$ . Let  $c$  be the critical number for  $f$ . How does the result of iterating Newton's method depend on the initial value chosen?

Now consider cubic polynomials. To simplify matters, note that if  $g(x) = kf(x)$  for some  $k$  then the iteration formula for Newton's method on  $g(x)$  is exactly the same as for  $f(x)$  so we need only consider cubics whose leading coefficient is 1.

2. Show that if  $g(x) = f(A(x))$  where  $A(x) = ax + b$ , then  $AN_gA^{-1}(x) = N_f(x)$ , where  $N_f(x)$  is the iteration formula for  $f$  and  $N_g(x)$  is the iteration formula for  $g$ .

This means that  $N_f(x)$  and  $N_g(x)$  are conjugate by an affine<sup>1</sup> transformation and thus exhibit the same dynamics; that is, if  $x_0$  is part of a periodic sequence (a sequence of iterates that repeat the same values over and over again) then  $A(x_0)$  is also part of a periodic sequence. That means that the affine transformation  $A$  doesn't change the dynamics of Newton's method.

To see a more detailed analysis of this, see the next page.

3. Let  $f(x) = (x+2)(x^2 + c)$  so that  $f$  is a cubic polynomial with a real zero at  $-2$  and a pair of real zeros at  $\pm\sqrt{-c}$  where  $c < 0$  is a negative real number. Simplify the Newton's method iteration formula  $N_f(x)$ , and find its critical points.
4. Analyze the dynamics of Newton's method for  $f(x) = (x+2)(x^2 + c)$  with  $c < 0$  for various initial values. How is the choice of the initial value related to the outcome of the iterative values?
5. Find a value of  $a$  so that  $N_f(a) = b$  and  $N_f(b) = a$  where  $a \neq b$  for various values of  $c < 0$ .
6. Investigate periodic cycles of  $N_f(x)$  for  $c > 0$ .

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<sup>1</sup> Affine: of, relating to, or being a transformation (as a translation, a rotation, or a uniform stretching) that carries straight lines into straight lines and parallel lines into parallel lines but may alter distance between points and angles between lines

Given any monic cubic,  $y = x^3 + kx^2 + mx + n$  we can shift it by  $h$  to obtain

$$\begin{aligned} y &= (x - h)^3 + k(x - h)^2 + m(x - h) + n \\ &= x^3 + (k - 3h)x^2 + (3h^2 - 2hk + m)x - h^3 + kh^2 - mh + n \end{aligned}$$

We'd like to choose  $h$  so that these coefficients match the pattern of  $y = (x - a)(x^2 + c)$  or,  $y = x^3 - ax^2 + cx - ac$ . Note that the constant term is the product of the coefficients of  $x$  and  $x^2$ .

If the cubic has three real zeros then, thinking about it graphically, it's obvious that such shifts exists: shift the graph so that two of the three real zeros are equidistant from  $x = 0$ . Evidently, three such shifts are possible: a shift to the midpoint between the first and second zeros, a shift to the midpoint between the second and third zeros or a shift to the midpoint between the first and third zeros.

At my advanced age I feel it appropriate to use a Computer Algebra System (CAS) to work the formulas. In Mathematica, enter  $a(x - h)^2 + b(x - h) + c + (x - h)^3$  which you can Expand[%] to get  $ah^2 - 2ahx + ax^2 - bh + bx + c - h^3 + 3h^2x - 3hx^2 + x^3$ , regroup with Apart[%] for  $x(-2ah + b + 3h^2) + ah^2 - x^2(3h - a) - bh + c - h^3 + x^3$ .

Now we want to find  $h$  so that the product of the coefficients of  $x$  and  $x^2$  equals the constant term: Simplify[ $(-2ah + b + 3h^2)(3h - a) + (-bh + c - h^3 + ah^2)$ ]

yields  $2a^2h - a(b + 8h^2) + 2bh + c + 8h^3$  which we can solve for  $h$  with Solve[% == 0, h]

This produces the usual staggering radical expressions (separated by { }) for the solutions of a cubic:

$$\begin{aligned} \{h \rightarrow & \frac{\sqrt[3]{-2a^3 + 3\sqrt{3}\sqrt{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2 + 9ab - 27c}}}{6\sqrt[3]{2}} \\ & - \frac{48b - 16a^2}{482^{2/3}\sqrt[3]{-2a^3 + 3\sqrt{3}\sqrt{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2 + 9ab - 27c}}} + \frac{a}{3}, \{h \\ \rightarrow & - \frac{(1 - i\sqrt{3})\sqrt[3]{-2a^3 + 3\sqrt{3}\sqrt{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2 + 9ab - 27c}}}{12\sqrt[3]{2}} \\ & + \frac{(1 + i\sqrt{3})(48b - 16a^2)}{962^{2/3}\sqrt[3]{-2a^3 + 3\sqrt{3}\sqrt{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2 + 9ab - 27c}}} + \frac{a}{3}, \{h \\ \rightarrow & - \frac{(1 + i\sqrt{3})\sqrt[3]{-2a^3 + 3\sqrt{3}\sqrt{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2 + 9ab - 27c}}}{12\sqrt[3]{2}} \\ & + \frac{(1 - i\sqrt{3})(48b - 16a^2)}{962^{2/3}\sqrt[3]{-2a^3 + 3\sqrt{3}\sqrt{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2 + 9ab - 27c}}} + \frac{a}{3} \} \end{aligned}$$

This makes it appear as if there's only one real solution, but what about looking at some particular cases?

What if we start with  $x^3 - 7x^2 + 14x - 8$ ? Substituting  $a = -7$ ,  $b = 14$  and  $c = -8$  into  $2a^2h - a(b + 8h^2) + 2bh + c + 8h^3 = 0$  yields Solve[ $98h + 7(14 + 8h^2) + 28h - 8 + 8h^3 == 0, h$ ] gives solutions  $\{h \rightarrow$

$-3, \{h \rightarrow -\frac{5}{2}\}, \{h \rightarrow -\frac{3}{2}\}$ .

These can be illustrated using Mathematica's Plot[] command as follows:

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Plot[{x^3 - 7x^2 + 14x - 8, (x + 3)^3 - 7(x + 3)^2 + 14(x + 3) - 8, (x + 2.5)^3 - 7(x + 2.5)^2 + 14(x + 2.5) - 8, (x + 1.5)^3 - 7(x + 1.5)^2 + 14(x + 1.5) - 8}, {x, -3, 5}, PlotRange -> 2.5, PlotStyle -> {Directive[Red, Thick], Directive[Blue, Thin], Directive[Green, Thick, Dashed], Directive[Brown, Thick]}
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