Magic Knight's Tours

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In *Mathematical Magic Show* [**4**], Martin Gardner looked at the classic problem of the knight's tour on a chess board, paying particular attention to tours in which the numbers in each row and column added to 260. At that time, the question of whether the numbers in each long diagonal could also be made to add to 260 was still open. Thanks to advances in computer power since Martin wrote, this question has now been decided, and perhaps readers will be interested in the complete statement that it is now possible to make.

A brief history of magic knight's tours

In knight's tour literature, the term *magic* is used for all tours in which the rows and columns add to the same number. Any diagonal or other properties are a bonus. Figure 1a is an example. This is a *closed* tour (the starting and finishing squares are a knight's move apart), and it has two-fold rotational symmetry (numbers which are diametrically opposite differ by 32, so if we draw the lines joining each number to the next, and 64 to 1, we get a pattern which repeats itself on rotation through 180 degrees). No knight's tour on an 8×8 board can be laterally or diagonally symmetric, nor can it have more than two-fold rotational symmetry, so Figure 1a has as much symmetry as we can

2	11	58	51	30	39	54	15	55	30	17	42	27	36	15	38
59	50	3	12	53	14	31	38	18	43	54	29	16	39	26	35
10	**1**	52	57	40	29	16	55	31	56	41	20	33	28	37	14
49	60	9	4	13	56	37	32	44	19	32	53	40	13	34	25
64	5	24	45	36	41	28	17	57	2	45	8	21	**64**	51	12
23	48	61	8	25	20	33	42	46	5	60	**1**	52	9	24	63
6	63	46	21	44	35	18	27	3	58	7	48	61	22	11	50
47	22	7	62	19	26	43	34	6	47	4	59	10	49	62	23

(a) Wenzelides, 1849 (b) Jaenisch, 1859

Figure 1. Two magic knight's tours, with starting and finishing squares in bold.

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hope for. We may notice that the numbers along a row or column are alternately odd and even.

The first magic knight's tour to appear in print was published by William Beverley in 1848, but this was an *open* tour (there was no knight's move from the final square back to the first square). Figure 1a, which was found by Karl Wenzelides and appeared in *Schachzeitung* in 1849, was the first closed magic knight's tour [**12**, **18**]. Several more closed magic tours appeared during the next few years, including Figure 1b, by C. F. Jaenisch, also rotationally symmetric, which appeared in *The Chess Monthly* in 1859 [**5**] and was described by its discoverer as "La solution la plus parfaite du problème du cavalier" ([**5**] is presented bilingually, but the French was the author's original). Although in neither of these tours do the long diagonals add to 260, the two long diagonals together do add to 520, and Figure 1b has a further property at which we look in a moment.

By 1940, 126 magic tours had been discovered, 59 of them closed. They were listed in 1951 in an unpublished monograph *The Magic Knight's Tours, a Mathematical Recreation* by H. J. R. Murray [**9**, **16**], but there was no convenient list in print until George Jelliss gave one in *Chessics* **26** in 1986 [**7**, **8**]. He accompanied it with a note that with modern computer methods it should be possible to ascertain whether all such tours had been discovered, and T. W. Marlow immediately showed that they hadn't by discovering five more, which he published in 1987 and 1988 [**14**, **15**]. T. S. Roberts found two more in January 2003 [**10**]. Still, none had proved to be diagonally magic, and a distributed computing exercise was set up by Hugues Mackay, Jean-Charles Meyrignac, and Guenter Stertenbrink later in 2003 to put the matter finally to rest. This exercise rediscovered all the magic tours previously known and added seven new ones, three open and four closed, giving a grand total of 140 (63 closed), none diagonally magic [**11**, **17**]. These conclusions were subsequently verified by Yann Denef using a program written independently [**17**].

The complete catalogue is on at least two web sites [**11**, **17**] and in *Variant Chess* **57** [**1**, **2**]. Each tour can be presented in any of eight orientations and can be numbered from either end, so these 140 tours give rise to 2240 different arithmetical matrices.

Job done?

Not quite. Only the 8×8 case was settled. We can ignore boards of odd side, because the row sums of a tour on such a board will be alternately odd and even, and so cannot all be the same. We can also ignore boards of side $4n + 2$, since George Jelliss has produced an elegant and ingenious proof implying that no magic tour on such a board exists [**11**]. Proofs in this field tend to be either elementary or non-existent, but this is an exception.

Theorem (G. P. Jelliss). *A magic knight's tour on a board of side* $4n + 2$ *is impossible.*

Lemma. In a magic knight's tour, the number of entries of the form $4k + 2$ or $4k + 3$ in any row or column, counted together, must be even.

Proof of the lemma. Let h be half the length of the side. The row sum is now $4h^3 + h$. If we express the numbers in the row in binary, we see that h of them, being odd, have a non-zero units digit, and adding these digits gives h. Remove these units digits from the entries, and the numbers represented by the remaining digits must sum to $4h^3$. This is a multiple of four and the units digits in the contributing numbers have been removed, so the number of non-zero twos digits in them must be even. But each entry of the form $4k + 2$ or $4k + 3$ contributes one such digit and no entry of the form $4k$ or $4k + 1$ contributes any, so the result follows. A similar argument applies to the columns. ■

Proof of the theorem. Label the squares of the chessboard thus:

$$\begin{array}{ccccccc} A & B & A & B & A & B & \dots \\ C & D & C & D & C & D & \dots \\ A & B & A & B & A & B & \dots \\ C & D & C & D & C & D & \dots \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots} \end{array}$$

We reflect the tour, if necessary, so that the number in the top left corner is odd; then all the A and D squares will contain odd numbers, and all the B and C squares even numbers. Now consider the distribution of the entries of the form $4k + 3$. There are $4n^2 + 4n + 1$ of these, which is an odd number, and they must all be in A or D squares, so either the A squares contain an odd number and the D squares an even number, or vice versa. Similarly, either the B squares contain an odd number of entries of the form $4k + 2$ and the C squares an even number, or vice versa.

Now suppose first that the A squares contain an odd number of $4k + 3$ entries and the B squares an odd number of $4k + 2$ entries; then the C squares will contain an even number of $4k + 2$ entries, and the total number of $4k + 3$ and $4k + 2$ entries on A and C squares, counted together, will be odd. But these squares appear only in alternate columns, and by the lemma each individual column must contain an even number of such entries. This is a contradiction, so the situation is impossible. A similar argument applies whichever of the A or D squares contain the odd number of $4k + 3$ entries and whichever of the B or C squares the odd number of $4k + 2$ entries.

Thus there can be no magic knight's tour on a board of side $4n + 2$. ■

So only boards of side $4n$ need be considered. T. H. Willcocks had discovered an open magic tour on a 12×12 board in which one diagonal added to the magic constant, and in April 2003 Awani Kumar found four such tours with both diagonals magic [**10**, **17**]. As far as I know, the existence of closed magic tours on this board with both diagonals also magic remains an open question, but two such tours on a 16×16 board were discovered by H. E. de Vasa and published in 1962 [**3**, **11**], and others have been discovered since [**13**, **17**].

Return to the 8 × 8 board

In the field of ordinary magic squares, where the numbers are not constrained to form a knight's tour, particular attention is paid to *pan-diagonal* squares, whose numbers add to the magic constant not only along rows, columns, and principal diagonals, but also along broken diagonals (in algebraic chess notation, diagonals such as a4-e8/f1-h3). Even while the possibility of a magic knight's tour with each long diagonal also adding to 260 was open, it was realised that no pan-diagonal tour could exist, because the numbers on a diagonal are either all odd or all even, and the 32 odd numbers from 1 to 64 don't add to 4×260; they add to 4×256, and the 32 even numbers to 4×264. So while preparing the catalogue of magic tours for *Variant Chess* **57**, I looked to see if any had a property which I called *quasi-magic*, where the odd diagonals, principal and broken, each add to 256, and the even diagonals add to 264. Given the inherent disparity between odd and even diagonals, this seemed to be as close to a truly pan-diagonal magic knight's tour as we could hope for.

It turned out that none had this property in full, but Figure 1b is half-way there. The principal odd diagonal adds to 256, as do each of the three *parallel* odd broken diagonals, and the principal even diagonal and the three *parallel* even broken diagonals each add to 264. The same is true of two further tours which Jaenisch published in 1862 [**6**], but those tours lack the rotational symmetry of Figure 1b. Because of this combination of properties, therefore, Figure 1b is indeed the "most perfect" solution to the problem of the knight's tour on the 8×8 board.

Now comes a curious question: was Jaenisch aware of this property of the broken diagonals in Figure 1b, which to my mind adds greatly to the elegance of this tour? I can find no mention in any of the sources I have examined from the nineteenth to the twenty-first century. The property will have become obvious as soon as it occurred to somebody to look for it, and I wrote in 2008 that I could not believe it had remained unspotted until then. Even so, nobody has yet drawn my attention to an earlier report of its existence.

Perhaps a reader of this JOURNAL will be able to enlighten me.

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Summary. The topic of the magic knight's tour, discussed by Martin Gardner in one of his books, is here brought up to date in the light of modern computer discoveries.

References

1. J. D. Beasley, Another look at 8×8 magic knight's tours, *Variant Chess* **57** (2008) 50–53.
2. ———, Credit where credit is due, *Variant Chess* **58** (2008) 72.
3. G. D'Hooghe, *Les Secrets du Cavalier*, Brepols, Bruxelles, 1962.
4. M. Gardner, Knights of the square table, in *Mathematical Magic Show*, George Allen and Unwin, London, 1977, 188–202, 283.
5. C. F. Jaenisch, De la solution la plus parfaite du problème du cavalier, *The Chess Monthly* (1859) 110–115, 146–151, 176–179.
6. ———, *Traité des Applications de l'Analyse Mathématique au Jeu des Échecs*, vol. 2, St Petersburg, 1862.
7. G. P. Jelliss, Catalogue of 8×8 magic knight's tours, *Chessics* **26** (1986) 122–128.
8. ———, Notes on Chessics 26, *Chessics* **29/30** (1987) 163.
9. ———, H. J. R. Murray's history of magic knight's tours, *The Games and Puzzles Journal* **14** (1996) 238–244, **15** (1997) 266–267.
10. ———, Recent advances in magic knight's tours, *Variant Chess* **43** (2003) 40–41.
11. ———, Knight's tour notes (2001–2005); available at http://www.mayhematics.com.
12. D. E. Knuth, letter to G. P. Jelliss, quoted in *The Games and Puzzles Journal* **14** (1996) 243.
13. A. Kumar, Studies in magic tours of knight on 16×16 board, *Journal of Recreational Mathematics* **34** (2005–2006) 275–285.
14. T. W. Marlow, Magic knight tours, *The Games and Puzzles Journal* **1** (1987) 11.
15. ———, Magic knight tours, *The Problemist* **12** (1988) 379.
16. H. J. R. Murray, *The Magic Knight's Tours, a Mathematical Recreation*, manuscript in the Bodleian Library, shelfmark MS Eng d.2370.
17. G. Stertenbrink, Computing magic knight's tours; available at http://magictour.free.fr.
18. K. Wenzelides (as "...l ....s in P......."), Bemerkungen über den Rösselsprung, *Schachzeitung* **4** (1849) 41–97.