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Path Representation of a Free Throw Shooter's Progress

CHRISTOPHER L. BOUCHER
Salem State College
Salem, MA 01970
cboucher@salemstate.edu

A problem on the Putnam exam given December 7, 2002 involved a basketball player (Shanille O'Keal) taking a sequence of free throws. The player makes the first shot, misses the second, and makes each subsequent shot with probability equal to the fraction of successful shots prior to that point. Thus, Shanille makes her third shot with probability $1/2$. If she makes her third shot, she makes her fourth shot with probability $2/3$ and so forth. The exam asked for the probability that she made 50 of her first 100 shots. We are interested in the probability that Shanille ever finds herself having missed k more shots than she has made.

Shanille's "state" after n shots can be represented by a pair (x, y) where x is the number of successful shots to that point in the sequence and y is the number of unsuccessful shots to that point in the sequence (so $n = x + y$). If Shanille's current state is (x, y), her history can be represented by a lattice path from $(1, 1)$ to (x, y) involving only rightward and upward steps where a rightward step represents a successful shot and an upward step an unsuccessful shot. Each edge in the lattice is naturally associated with a conditional probability . The horizontal edge connecting (x, y) and $(x + 1, y)$ has probability equal to the probability that the Shanille, having made x shots and missed y shots, makes her next shot. According to our rule, this probability is $x/(x + y)$. Similarly, the vertical edge connecting (x, y) and $(x, y + 1)$ has probability $y/(x + y)$. The probability of her history following any particular lattice path is the product of the probabilities associated with the edges of the path.

THEOREM 1. *The two lattice paths connecting (x, y) and $(x + 1, y + 1)$ are equiprobable.*

Proof. If we write RU for the path from (x, y) to $(x + 1, y + 1)$ consisting of a rightward step followed by an upward step and UR for the path from (x, y) to $(x + 1, y + 1)$ consisting of an upward step followed by a rightward step, then

$$P(RU) = \frac{x}{x+y} \cdot \frac{y}{x+y+1}$$

$$= \frac{y}{x+y} \cdot \frac{x}{x+y+1}$$
$$= P(UR). \qquad \blacksquare$$

It follows from Theorem 1 that if one path can be obtained from another by transposing two steps, then the two paths are equiprobable. Since any permutation (i.e., a reordering of the steps) of a path can be accomplished by a sequence of transpositions, all permutations of a path are equiprobable, a fact we record in this corollary.

COROLLARY 1. *Let α be a lattice path from $(1, 1)$ to (x, y) consisting of $x - 1$ rightward and $y - 1$ upward steps and let $\sigma(\alpha)$ be a permutation of α. Then α and $\sigma(\alpha)$ are equiprobable.*

In particular, Corollary 1 implies that any two lattice paths with the same number of rightward and upward steps are equiprobable. Thus, to calculate the probability that Shanille arrives at any point (x, y) we need only calculate the probability of any one path consisting of $x - 1$ rightward and $y - 1$ upward steps from $(1, 1)$ and count how many such paths there are. The path consisting of $x - 1$ consecutive rightward steps from $(1, 1)$ to $(x, 1)$ has probability

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{x-1}{x} = \frac{1}{x}.$$

The path consisting of $y - 1$ consecutive upward steps from $(x, 1)$ to (x, y) has probability

$$\frac{1}{x+1} \cdot \frac{2}{x+2} \cdots \frac{y-1}{x+y-1} = \frac{x!\,(y-1)!}{(x+y-1)!}.$$

We record these calculations in the following theorem.

THEOREM 2. *The lattice path from $(1, 1)$ to (x, y) consisting of $x - 1$ rightward steps followed by $y - 1$ upward steps has probability*

$$p(x, y) \doteq \frac{(x-1)!(y-1)!}{(x+y-1)!}.$$

The number of lattice paths from $(1, 1)$ to (x, y) is

$$\binom{x+y-2}{x-1}$$

and so the probability that Shanille winds up at state (x, y) is, remarkably,

$$\binom{x+y-2}{x-1} \frac{(x-1)!\,(y-1)!}{(x+y-1)!} = \frac{1}{x+y-1}.$$

In particular, the $n - 1$ states in which Shanille could be after taking n shots are all equiprobable and the expected number of shots made in n attempts is $n/2$. (Thus the solution to the Putnam question is $1/(100 - 1) = 1/99$.)

The use of lattice paths to visualize the history of our basketball player caused me to read with interest **[1]**, which calculates the probability that a one dimensional random walk returns to the origin given that the walker starts at $x = k$ by viewing the walker's progress as a lattice path. A similar analysis can be applied to our basketball player. For $k \in \mathbb{N}$, let P_k be the probability that our shooter ever finds herself having missed k more shots than she has made, that is, the probability that she ever finds herself at a state of the form $(n, n+k)$ for some $n \in \mathbb{N}$. This is the analog of the probability addressed in [**1**]. The number of $2n+k-2$-length lattice paths from $(1, 1)$ to $(n, n+k)$ that intersect the line $y = x + k$ only at the point $(n, n+k)$ is the same as the number of $2n+k-2$-length lattice paths from (0, 0) to $(n-1, n-1+k)$ that intersect the line $y = x+k$ only at the point $(n-1, n-1+k)$. The article [**1**] denotes this number $C_k(n-1)$, where

$$C_1(n-1) = \frac{1}{n}\binom{2n-2}{n-1}$$

is the $(n-1)$st Catalan Number, $C_2(n-1) = C_1(n)$, and for $k \geq 3$, the numbers $C_k(n)$ satisfy the following recurrence relation (see Theorem 2 in [**1**]):

$$C_k(n) = C_{k-1}(n+1) - C_{k-2}(n+1). \tag{1}$$

The probability we seek is

$$P_k = \sum_{n=1}^{\infty} C_k(n-1)p(n, n+k). \tag{2}$$

The analogous sum in [**1**] is different because for the walker, all steps up are equiprobable and all steps right are equiprobable, while for the basketball player it is paths with the same initial and terminal point that are equiprobable. If $k = 1$, then using

$$C_1(n-1) = \frac{1}{n}\binom{2n-2}{n-1},$$

we have

$$\begin{aligned} P_1 &= \sum_{n=1}^{\infty} C_1(n-1)p(n, n+1) = \sum_{n=1}^{\infty} \frac{1}{(2n)(2n-1)} \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2n-1}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2. \end{aligned} \tag{3}$$

So the probability that, at some point, Shanille has missed one more shot that she has made is, remarkably, $\log 2$. (Conversely, the probability that she has always made at least as many as she has missed is $1 - \log 2$.) For $k = 2$, we have

$$P_2 = \sum_{n=1}^{\infty} C_2(n-1)p(n, n+2) = \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - 2\log 2. \tag{4}$$

An evaluation of the sum in (2) for general k, however, requires a closed form expression for $C_k(n)$, which is given in the following theorem.

THEOREM 3. *If* $n, k \in \mathbb{N}$ *then*

$$C_k(n) = \frac{k}{2n+k}\binom{2n+k}{n}. \tag{5}$$

Proof. For $n = 1, 2\ldots$ and $k = 0, 1, \ldots$, let

$$B_k(n) = \frac{k}{2n+k}\binom{2n+k}{n},$$

and let $C_k(n)$ be the numbers that satisfy (1) with $C_0(n) = 0$ and

$$C_1(n) = \frac{1}{n+1}\binom{2n}{n}.$$

(Note that this implies that $C_2(n) = C_1(n+1)$). For $n \in \mathbb{N}$, $B_0(n) = C_0(n)$ and

$$B_1(n) = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{2n+1} \cdot \frac{(2n+1)!}{n!\,(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = C_1(n).$$

Using similarly straightforward algebra, one can verify that for all $n = 1, 2, \ldots$ and $k = 0, 1, \ldots$ the numbers $B_k(n)$ satisfy:

$$B_k(n) = B_{k-1}(n+1) - B_{k-2}(n+1). \tag{6}$$

This is the same recurrence relation satisfied by the numbers $C_k(n)$. It follows that the numbers $B_k(n)$ are identical to the numbers $C_k(n)$, which is what the theorem asserts. ■

Having a closed form expression for the path counts doesn't appear to be much use in computing the probability that the walker ever finds himself k steps to one side of where he started, however it does allow us to solve the analogous problem for Shanille, i.e., to compute (2) for general k.

$$\begin{aligned} P_k &= \sum_{n=1}^{\infty} C_k(n-1)p(n, n+k) \\ &= \sum_{n=1}^{\infty} \frac{k}{2(n-1)+k}\binom{2(n-1)+k}{n-1}\frac{(n-1)!\,(n+k-1)!}{(2n+k-1)!} \\ &= \sum_{n=1}^{\infty} \frac{k}{(2n+k-2)(2n+k-1)} \\ &= \sum_{n=1}^{\infty} k\left(\frac{1}{2n+k-2} - \frac{1}{2n+k-1}\right) \\ &= k\left[\frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+2} - \frac{1}{k+3} + \cdots\right] \\ &= k \cdot (-1)^{k+1}\left(\log 2 - \sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{j}\right). \end{aligned} \tag{7}$$

The last line in this display uses the series

$$\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Note that the sums in (3) and (4) are computed in the same manner as we compute P_k here. The first ten values of P_k are displayed below in Table 1.

The numerical evidence suggests that these probabilities decline from $\log 2$ to $1/2$. This is in fact the case. Since the event that Shanille ever finds herself having missed k more shots than she has made is a superset of the event that she ever finds herself having missed $k+1$ more shots than she has made, the probabilities certainly decline.

TABLE 1: Some values of P_k

k	P_k
1	.693147
2	.613706
3	.579442
4	.560745
5	.549069
6	.541117
7	.535364
8	.531013
9	.527610
10	.524877

Thus, $\lim_{k\to\infty} P_k$ surely exists. Determining the value of this limit is a nice exercise in several techniques from first-year calculus. First, we note that if

$$f_k(x) = \frac{k}{(2x+k-2)(2x+k-1)},$$

then

$$f_k'(x) = -\frac{2k(4x+2k-3)}{(2x+k-2)^2(2x+k-1)^2}$$

and for any $k = 1, 2, \ldots$, f_k is a decreasing function on $[1, \infty)$. Thus, we can use integrals to bound the series (7) above and below:

$$\int_1^\infty f_k(x)\,dx \le \sum_{n=1}^{\infty} f_k(n) \le f_k(1) + \int_1^\infty f_k(x)\,dx. \tag{8}$$

Next, we calculate the improper integrals above:

$$\int_1^\infty f_k(x)\,dx = \frac{k}{2}\log\left(1+\frac{1}{k}\right)$$

So, we have for any $k = 1, 2, \ldots$,

$$\frac{k}{2}\log\left(1+\frac{1}{k}\right) \le \sum_{n=1}^{\infty} f_k(n) \le \frac{k}{k(k+1)} + \frac{k}{2}\log\left(1+\frac{1}{k}\right).$$

Finally, an application of L'Hôpital's Rule shows that

$$\lim_{k\to\infty} \frac{k}{2}\log\left(1+\frac{1}{k}\right) = \frac{1}{2}.$$

This proves our claim that the probability that Shanille ever finds herself having missed k more shots than she has made approaches $1/2$ as $k \to \infty$.

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